

PLANE JACOBIAN CONJECTURE FOR RATIONAL POLYNOMIALS

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ABSTRACT. A non-zero constant Jacobian polynomial maps $F = (P, Q)$ of \mathbb{C}^2 is invertible if P and Q are rational polynomials.

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1. We shall call a polynomial map $F = (P, Q) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ is *Keller map* if F satisfies the Jacobian condition $J(P, Q) := P_x Q_y - P_y Q_x \equiv c \neq 0$. The Jacobian conjecture, posed first in 1939 by Ott-Heinrich Keller [8] and still opened, asserts that *every Keller map is invertible*. We refer the readers to the nice surveys [3] and [4] for the history, the recent developments and the related topics of this mysterious problem.

One of most simple topological cases of Keller's question, for which we may hope to have a complete solution by using the rich knowledge on plane algebraic curves, is the case when one or both of P and Q are *rational polynomials*, i.e. polynomials with generic fibre diffeomorphic to the sphere with finite number of punctures. Since 1978 Razar had found the following.

Theorem 1 (Razar's Theorem, [22]). *A Keller map $F = (P, Q)$ is invertible if P is a rational polynomial with all irreducible fibres.*

In attempting to understand the nature of the plane Jacobian conjecture, Razar's theorem had been reproved by Heitmann [6], Lê and Weber [12], Friedland [21], Nemethi and Sigra [14] in several different algebraic and algebro-geometric approaches. In fact, Vistoli [23] and Neumann and Norbury [15] observed that every rational polynomial with all irreducible fibres is equivalent to the projection $(x, y) \mapsto x$ up to algebraic coordinates. Recently, Lê ([9], [10]) proved that a Keller map $F = (P, Q)$ is invertible if P is a rational polynomial and, in addition, P is simple, i.e. in regular extensions $p : X \longrightarrow \mathbb{P}^1$ of P over a compactification X of \mathbb{C}^2 the restriction of p to each irreducible component of the divisor at infinity $\mathcal{D} := X \setminus \mathbb{C}^2$ have degree 0 or 1. As shown in [20], Lê's result is still true without the condition that P is rational.

In this paper we would like to note that the plane Jacobian conjecture is true for the case when both of P and Q are rational.

Theorem 2 (Main Theorem). *Suppose $F = (P, Q)$ is a Keller map. If P and Q are rational polynomials, then F is invertible.*

By a *polar branch* we mean an irreducible branch curve at infinity along which F tends to infinity. An obvious simple fact is that under the Jacobian condition any irreducible component of any fiber of P must contains some polar branches.

Otherwise, the restriction of F to such an exceptional component must be constant mapping that is impossible. In order to prove Theorem 2 we will try to show that each of fibres of P has only one polar branch. This ensures that all fibres of P are irreducible. Then, invertibility of F follows from Razar's theorem.

2. Our proof is based on the following facts on Keller maps.

Following [7], the *non-proper value set* A_f of a polynomial map f of \mathbb{C}^2 is the set of all values $a \in \mathbb{C}^2$ such that $f(b_i) \rightarrow a$ for a sequence $b_i \rightarrow \infty$. This set A_f is either empty or an algebraic curve in \mathbb{C}^2 each of whose irreducible components is the image of a non-constant polynomial map from \mathbb{C} into \mathbb{C}^2 . If f is a Keller map, the restriction

$$f : \mathbb{C}^2 \setminus f^{-1}(A_f) \longrightarrow \mathbb{C}^2 \setminus A_f$$

gives a unbranched covering.

Theorem 3 (see Theorem 4.4 in [16]). *Suppose $F = (P, Q)$ is a Keller map. The non-proper value set A_F , is not empty, is composed of the images of some polynomial maps $t \mapsto (\alpha(t), \beta(t))$, $\alpha, \beta \in \mathbb{C}[t]$, satisfying*

$$\frac{\deg \alpha}{\deg \beta} = \frac{\deg P}{\deg Q}. \quad (1)$$

In particular, A_F can never contains smooth irreducible components to \mathbb{C} .

This fact was presented in [16] and can be reduced from [2] (see also [18] and [19] for other refine versions). It can be used in assuming that there is a plane algebraic curve E , composed of some irreducible components parameterized polynomial maps $(\alpha(t), \beta(t))$ satisfying (1), such that the restriction

$$F : \mathbb{C}^2 \setminus F^{-1}(E) \longrightarrow \mathbb{C}^2 \setminus E \quad (2)$$

gives a unbranched covering. Each of irreducible components of such a curve E is a singular curve approaching to $(\infty, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1 \supset \mathbb{C}^2$. In working with generic fibres of P we may use the following convenience, which was presented in [17] in a little different statement.

Proposition 1 (See Theorem 1 in [17]). *Suppose $F = (P, Q)$ is a Keller map and E is a plane algebraic curve in (2). If the line $L_c := \{(c, t) : t \in \mathbb{C}\}$ intersects transversally each irreducible component of the non-proper set E , then $P = c$ is a generic fiber of P .*

3. Below we shall reduce from the results introduced in the previous section some advance estimations on the polar branches and the genus of generic fibres of P and Q

For each $c \in \mathbb{C}$ let us denote by $Pol(P, c)$ the union of polar branches in the fiber $P = c$. The germ curve $Pol(P, c)$ can be realized as the inverse image $F^{-1}(V(c, R))$ for enough large number $R > 0$, where $V(c, R) := \{(c, t) : |t| > R\}$.

Theorem 4. *Suppose $F = (P, Q)$ is a Keller map. Then, the family of germ curves $Pol(P, c)$, $c \in \mathbb{C}$, is equianalytical. Namely, for each $c_0 \in \mathbb{C}$ there exists a small dick $\Delta \subset \mathbb{C}$ centered at c_0 and a number $R > 0$ such that the restriction*

$$P : \bigcup_{c \in \Delta} F^{-1}(V(c, R)) \longrightarrow \Delta$$

is an analytic fibration. In particular, the number of irreducible components in $\text{Pol}(P, c)$ does not depend on $c \in \mathbb{C}$.

This fact was contained implicitly in Theorem 3.4 in [16], stated in terms of Newton-Puiseux expansions. For convenience, we present here a proof by applying Theorem 3.

Proof of Theorem 4. In view point of Theorem 3 we can assume that there is a plane algebraic curve E , composed of some irreducible components parameterized by polynomial maps $(\alpha(t), \beta(t))$ with

$$\frac{\deg \alpha}{\deg \beta} = \frac{\deg P}{\deg Q},$$

such that the restriction

$$F : \mathbb{C}^2 \setminus F^{-1}(E) \longrightarrow \mathbb{C}^2 \setminus E$$

gives a unbranched covering.

Let $c_0 \in \mathbb{C}$ be given. Since the components of E approach to the point $(\infty, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1 \supset \mathbb{C}^2$, there exists a small disk $\Delta \subset \mathbb{C}$ centered at c_0 and a number $R > 0$ such that

$$(\Delta \times \{t \in \mathbb{C} : |t| > R\}) \cap E = \emptyset.$$

This ensures that the restriction

$$F : F^{-1}(\Delta \times \{t \in \mathbb{C} : |t| > R\}) \longrightarrow \Delta \times \{t \in \mathbb{C} : |t| > R\}$$

is a unbranched analytic covering. Note that

$$F^{-1}(\Delta \times \{t \in \mathbb{C} : |t| > R\}) = \cup_{c \in \Delta} F^{-1}(V(c, R)).$$

This follows that

$$P : \bigcup_{c \in \Delta} F^{-1}(V(c, R)) \longrightarrow \Delta$$

is an analytic fibration. \square

Let us denote by g_h the genus of the generic fiber of a primitive polynomial $h \in \mathbb{C}[x, y]$.

Lemma 1. *Suppose $F = (P, Q)$ is a Keller map. If $\deg P \leq \deg Q$, then $g_P \leq g_Q$.*

Proof. If F is invertible, of course $g_P = g_Q = 0$. Consider the situation when F is not invertible. We need prove only that if $\deg P \leq \deg Q$, then for each t enough small a generic fiber of P can be topologically embedded into a generic fiber of $P + tQ$. This ensures that $g_P \leq g_{P+tQ}$ for each t enough small that implies the desired conclusion.

Let A_F be the non-proper value set of F . Note that A_F is a curve in \mathbb{C}^2 and the restriction $F : \mathbb{C}^2 \setminus F^{-1}(A_F) \longrightarrow \mathbb{C}^2 \setminus A_F$ gives a unramified covering. Replacing F by $F + p$ for a generic point $p \in \mathbb{C}^2$ if necessary, we can assume that $(0, 0) \in \mathbb{C}^2 \setminus A_F$ and for $|t| < \epsilon$ the lines L_t given by $u + tv = 0$ intersects transversally A_F . By Proposition 1 the late ensures that for $|t| < \epsilon$ the curve $P + tQ = 0$ is a generic fiber of $P + tQ$.

Now, we will construct topological embeddings $(P = 0) \hookrightarrow (P + tQ = 0)$ for $|t| < \epsilon$. To do it, we can choose a box $B := \{|u| < r; |v| < s\}$ such that $(L_0 \cap A_F) \subset B$ and $A_F \cap B$ is a smooth manifold. Since the lines L_t intersects transversal A_F , by standard arguments we can modify the motion $\phi_t(0, v) := (-tv, v)$ such that

$\phi_t(A_F \cap L_0) \subset A_F \cap L_t$ and $\phi_t : L_0 \cap B \rightarrow \phi_t(L_0 \cap B) \subset L_t \cap B$ are diffeomorphisms. Let Φ_t be the lifting map Φ_t induced by the covering $F : \mathbb{C}^2 \setminus F^{-1}(A_F) \rightarrow \mathbb{C}^2 \setminus A_F$. Then, $\Phi_t : F^{-1}(L_0 \cap B) \rightarrow \{P + tQ = 0\}$ is an embedding of $F^{-1}(L_0 \cap B)$ into the fiber $P + tQ = 0$. Since $(L_0 \cap A_F) \subset B$, it is easy to see that the fiber $P = 0$ can be deformed diffeomorphic to its subset $F^{-1}(L_0 \cap B)$. So, we get the desired embeddings. \square

4. Now, we are ready to prove the main result.

Proof of Theorem 2. Let $F = (P, Q)$ be a given Keller map. Assume that P and Q are rational polynomials. We can assume in addition that the following conditions holds:

- a) $\deg P < \deg Q$;
- b) The curve $P = 0$ is irreducible;
- c) For generic $\lambda \in \mathbb{C}$ the curve $\lambda P + Q$ is generic fiber of $\lambda P + Q$.

Indeed, if $\deg P = \deg Q$, by the Jacobian condition we have $P_+ = cQ_+$ for a number $c \neq 0$, where P_+ and Q_+ are leading homogenous components of P and Q , respectively. Then, $\deg P - cQ < \deg Q$ and $P - cQ$ is also rational by Lemma 1. So, we can replace P by $P - cQ$. Further, in view of Theorem 3 and Proposition 1, we can choose a generic point $(a, b) \in \mathbb{C}^2$ such that the curve $P = a$ is a generic fiber of P and the curve $\lambda P + Q = \lambda a + b$ is those of $\lambda P + Q$ for generic $\lambda \in \mathbb{C}$. So, we can replace F by $F - (a, b)$.

We will show that the fiber $P = 0$ has only one polar branch. Then, by Theorem 5 every fiber of P also has only one polar branch. This ensures that all fibres of P are irreducible. Therefore, by Razar's theorem F is invertible.

To do it, we regard the plane \mathbb{C}^2 as a subset of the projective plane \mathbb{P}^2 and associate to $F = (P, Q)$ the rational map $G_F : \mathbb{P}^2 \rightarrow \mathbb{P}^1$ defined by $G_F(x, y) = Q(x, y)/P(x, y)$ on the part \mathbb{C}^2 . The indeterminacy point set of G_F consists of the set $B := F^{-1}(0, 0)$ and a finite number of points in the line at infinity L_∞ of the chart \mathbb{C}^2 . Note that the restriction of G_F to L_∞ is equal to the zero, since $\deg P < \deg Q$. By blowing-up we can remove indeterminacy points of G_F and obtain a blowing-up map $\pi : X \rightarrow \mathbb{P}^2$ and a regular extension $g_F : X \rightarrow \mathbb{P}^1$ over a compactification X of $\mathbb{C}^2 \setminus B$.

Now, suppose $g_F : X \rightarrow \mathbb{P}^1$ is such a regular extension of G_F . Let $\mathcal{D}_\infty := \pi^{-1}(L_\infty)$ and $\mathcal{D}_b := \pi^{-1}(b)$, $b \in B$. \mathcal{D}_∞ is a connected rational curve with simple normal crossing and its dual graph is a tree. Each \mathcal{D}_b is a copy of \mathbb{P}^1 . The divisor $\mathcal{D} := X \setminus (\mathbb{C}^2 \setminus B)$ then is a distinct union of \mathcal{D}_∞ and \mathcal{D}_b . Observe, for generic $\lambda \in \mathbb{P}^1$ the fiber $g_F = \lambda$ is the closure in X of the portion $\{(x, y) \in \mathbb{C}^2 \setminus B : \lambda P(x, y) + Q(x, y) = 0\}$ of the fiber $\lambda P + Q = 0$. Since $\deg P < \deg Q$ by Condition (a), in view of Lemma 1 the polynomials $\lambda P + Q$ are rational. Therefore, by Condition (c) generic fibres of g_F are irreducible rational curves. This means that $g_F : X \rightarrow \mathbb{P}^1$ is a \mathbb{P}^1 -fibration over \mathbb{P}^1 .

Now, we consider the fiber of g_F over ∞ , denoted by C_∞ . Let us denote $D_\infty := C_\infty \cap \mathcal{D}_\infty$ and by Γ the closure in X of the portion $\{(x, y) \in \mathbb{C}^2 \setminus B : P(x, y) = 0\}$. By the conditions (a) and (b) Γ is an irreducible rational curve and D_∞ contains at least the proper transform of the line at infinity of \mathbb{C}^2 . Furthermore, by the Jacobian condition the multiplicity of g_F on Γ is equal to 1. Obviously, $C_\infty = D_\infty \cup \Gamma$ and Γ intersects D_∞ at polar branches of the fiber $P = 0$. By the well-know fact (see

for example [5] and [1]), that any reducible fiber of a \mathbb{P}^1 -fibration over \mathbb{P}^1 can be contracted by blowing down to any its component of multiplicity 1, the fiber C_∞ can be contracted to Γ . In particular, D_∞ is a blowing-up version of one point and C_∞ is a blowing-up version of \mathbb{P}^1 . Hence, the dual graphs of C_∞ and D_∞ are tree. Therefore, the curve Γ intersects transversally D_∞ at a unique smooth point of D_∞ . This follows that the fiber $P = 0$ has only one polar branch. \square

5. To conclude we would like to note that it remains open the question *whether a Keller map $F = (P, Q)$ with rational component P is invertible*. In view of Theorem 2 and its proof, if such a Keller map is not invertible, then $\deg P < \deg Q$. In fact, it is possible to show that in such a Keller map $\deg P$ does not divide $\deg Q$ and the fibres of P have exactly two polar branches. We will return to discuss on the question in a further paper.

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